Seminar Notes

Many Body Localization

Recent Advances in Quantum Many Body Theory Seminar
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1 Introduction

In this talk the results from single-particle Anderson Localization are applied to a non-interacting fermionic many body system as a first step towards Many Body Localization. We'll find that in the non-interacting case, we can still find approximately localized eigenmodes that can be filled by fermions, much like electrons filling the orbitals of the hydrogen atom.

In the end I would like to give an outlook on more general systems, where we allow particles to interact.

2 Single Particle Anderson Results

Let us remember the single-particle Anderson setting and our main results from last week.

We have the typical Anderson Hamiltonian given as

$$H_A = -\sum_x |x\rangle\langle x+1| + |x+1\rangle\langle x| + \sum_x (v_0 + v_{\omega_x}) |x\rangle\langle x| \,, \tag{1}$$

where $\{\, |x\rangle \,\}$ denotes the *lattice basis*. The first sum is the hopping term and the second sum consists of an on-site potential v_0 and the typical random *i.i.d.* potential v_{ω_x}.

In that Basis the hamiltonian takes the following shape:

$$H_A = \begin{bmatrix} v_0 + v_{\omega_1} & -1 & 0 & \dots & & 0 \\ -1 & v_0 + v_{\omega_2} & -1 & & & \vdots \\ 0 & & \ddots & & & \\ \vdots & & & & & \\ & & & & & \\ 0 & \dots & & 0 & -1 & v_0 + v_{\omega_L} \end{bmatrix} \tag{2}$$

We know that the eigenvectors of such a system are *'easy'*. By easy we mean that their overlap with the lattice basis vectors is exponentially supressed by a uniform *localization length* l_0 and therefore only a few lattice basis vectors that are close to each other contribute to any given eigenvector. Put precisely we find that

$$\exists l_0 \, \forall \varphi_k \exists x_k : |\varphi_k(j)| \leq \mathrm{e}^{-|x_k - j|/l_0} \,, \tag{3}$$

where $|x_k - j|$ is the *distance* between lattice site x_k and j.

Precisely speaking, this result only holds almost surely (for almost all $\{\, v_{\omega_i} \,\}$) and only for infinite dimensional systems, but as physicists we'll assume that this also holds for sufficiently large finite systems.

3 Non-Interacting Many Body System

3.1 Fermionic 1D Chain

We would like a 1D lattice of size L. Our Hilbert space is given by $\mathcal{H} = \mathbb{C}^{2^L}$.

Let's put together a small tool box for our endeavour:

We denote our vacuum state as $|\emptyset\rangle = |000\ldots0\rangle$ and define fermionic creation and annihilation operators $f_j^\dagger$ and f_j acting on site j:

$$f_j^\dagger |\emptyset\rangle = |0\ldots 0\, 1_j\, 0\ldots\rangle =: |1_j\rangle\ . \tag{4}$$

We demand the usual properties:

- Anticommutator $\{f_j^\dagger, f_i^\dagger\} = 0$
- $f_j^\dagger f_j =: n_j$ number operator on j-th site
- $(f_j^\dagger)^2 = (f_j)^2 = 0$ as we don't want to allow for more than one particle on one site.
- $f_j |\emptyset\rangle = 0$
- $f_j^\dagger f_j + f_j f_j^\dagger = \mathbb{1}$

Let's write down our hamiltonian in this language, not allowing for interactions between particles and quietly dropping the on-site potential v_0, because it doesn't fundamentally affect our system:

$$H_{NI} = -\sum_j f_{j+1}^\dagger f_j + f_j^\dagger f_j\ + \sum_j v_{\omega_j}\, f_j^\dagger f_j \tag{5}$$

3.2 Eigenmodes of H_{NI}

Our goal now is to diagonalize the hamiltonian and show that the eigenmodes are approximately local.

Let's first rewrite the hamiltonian $H_{NI} = \mathbf{f}^\dagger h \mathbf{f}$ by collecting the creation and annihilation operators in vectors.

$$\begin{aligned} H_{NI} &= \begin{bmatrix} f_1^\dagger & f_2^\dagger & \ldots & f_L^\dagger \end{bmatrix} \begin{bmatrix} v_0 + v_{\omega_1} & -1 & 0 & \ldots & 0 \\ -1 & v_0 + v_{\omega_2} & -1 & & \vdots \\ 0 & & \ddots & & \\ \vdots & & & & \\ 0 & \ldots & 0 & -1 & v_0 + v_{\omega_L} \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \\ \vdots \\ \\ f_L \end{bmatrix} \\ &= \mathbf{f}^\dagger\, h\, \mathbf{f} \\ &= \mathbf{f}^\dagger\, U D U^\dagger\, \mathbf{f}\,, \end{aligned}$$

where $D = diag(\lambda_i)$ and U is the usual unitary. We notice that while the $f_j^\dagger$ still live in $\mathbb{C}^{2^L \times 2^L}$, the coefficient matrix h is only from $M^{L\times L}$ and we are familiar with it's shape, because it is similar to the standard Anderson hamiltonian H_A.

We can now write

$$H_{NI} = \tilde{\mathbf{f}}^\dagger D \,\tilde{\mathbf{f}} \tag{6}$$
$$= \sum_k \lambda_k \tilde{f}_k^\dagger \tilde{f}_k \,. \tag{7}$$

We chose $\tilde{\mathbf{f}^\dagger} = U^\dagger \mathbf{f}$ or

$$\begin{bmatrix} \tilde{f}_1 \\ \tilde{f}_2 \\ \vdots \\ \tilde{f}_L \end{bmatrix} = \begin{bmatrix} (\varphi_1(1) & \varphi_1(2) & \dots & \varphi_1(L)) \\ & \langle \varphi_2 | & & \\ & \vdots & & \\ & \langle \varphi_L | & & \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_L \end{bmatrix} \tag{8}$$

So we see that the k-th entry is given by

$$\tilde{f}_k = \sum_j \varphi_k(j)\, f_j \tag{9}$$

3.3 Approximate Locality

In order to show that the eigenmodes $\tilde{f}_k$ are approximately local, we first need to introduce a little bit of technique, because here the eigenmodes $\tilde{f}_k$ are operators and we only have a good idea of local wave functions so far.

Let's talk about bases first. On any single site j we can define a basis $\{ \mathbb{1}_j,\, f_j,\, f_j^\dagger,\, f_j^\dagger f_j \} =: \{ \gamma_j^1,\, \gamma_j^2,\, \gamma_j^3,\, \gamma_j^4 \}$, relabeling them as γ_j^i. A general operator on our system can be written as

$$A = \sum_\alpha^{4^L} c_\alpha \prod_j^L \gamma_j^i \,, \qquad i \in \{ 1, 2, 3, 4 \} \,, \tag{10}$$

where the product runs through all combinations of γ_j^i.

We define an operator to be *approximately local* $:\Longleftrightarrow$

$$\| A - \Gamma_{X_l}(A) \| \leq \mathrm{e}^{-l/l_0} \,. \tag{11}$$

Here the *reduction map* Γ_{X_l} maps an operator A to an operator A_{X_l} that is only affecting a sphere of size l of our lattice $X_l \subset \Lambda$, but is otherwise the same as A. Specifically this means that

$$\Gamma_{X_l}(A) = \sum_\alpha^{4^L} c_\alpha \prod_j^L \gamma_j^i \,, \qquad i = 1 \text{ for } j \in X_l^C \tag{12}$$

Trying to find that the $\tilde{f}_k$ are approximately local, we plug in:

$$\begin{aligned}
\|\tilde{f}_k - \Gamma_{X_l}(\tilde{f}_k)\| &= \| \sum_j \varphi_k(j)\, f_j - \sum_{j\in X_l} \varphi_k(j)\, f_j \| \\
&= \| \sum_{j\in X_l^C} \varphi_k(j)\, f_j \| \\
&\leq \sum_{j\in X_l^C} |\varphi_k(j)|\, \|f_j\| \\
&\leq \sum_{j\in X_l^C} |\varphi_k(j)| \\
&\leq \sum_{j\in X_l^C} \mathrm{e}^{-|x_k - j|/l_0}\,,
\end{aligned}$$

Having estimated $\|f_j\| \leq 1$ from above.

Splitting and shifting the sums we can write

$$\begin{aligned}
\|\tilde{f}_k - \Gamma_{X_l}(\tilde{f}_k)\| &\leq \sum_{j\in X_l^C} \mathrm{e}^{-|x_k - j|/l_0} && (13) \\
&= \sum_{l'\geq l} \left(\mathrm{e}^{-l'/l_0} \sum_{\{\, j:|x_k - j| = l'\,\}} \right) \\
&= \mathrm{e}^{-l/l_0} \sum_{l'\geq 0} 2\, \mathrm{e}^{-l'/l_0} \\
&= \mathrm{e}^{-l/l_0}\ 2 \sum_{l'\geq 0} \left(\frac{1}{e^{1/l_0}} \right)^{l'} \\
&= \mathrm{e}^{-l/l_0}\ \frac{2}{1 - \mathrm{e}^{-1/l_0}} \\
&= c\, \mathrm{e}^{-l/l_0} \\
&= \mathrm{e}^{-(l - l_0 \ln c)/l_0)} \\
\|\tilde{f}_k - \Gamma_{X_l}(\tilde{f}_k)\| &\leq \mathrm{e}^{-\tilde{l}/l_0} && (14)
\end{aligned}$$

We recognized the geometric series, which made this very simple. However, for any dimension where φ_k is local, we can find a respective constant c as an upper bound to a more complex polynomial in place of the geometric series.

4 Results & Outlook

We have shown that the non-interacting many body hamiltonian has approximately local eigenmodes.

We can indeed think of this system the same way we think about the hydrogen atom: as filling eigenmodes by consecutively putting electrons into them.

The non-interacting model hamiltonian

$$\begin{aligned} H_{NI} &= -\sum_j f_{j+1}^\dagger f_j + f_j^\dagger f_j \; + \sum_j v_{\omega_j}\, f_j^\dagger f_j \\ &= \sum_k \lambda_k \tilde{f}_k^\dagger \tilde{f}_k \end{aligned}$$

however, has it's limitations and we cannot generalize our results to systems where fermion-fermion interaction is present.

Outlook

From the locality of the non-interacting eigenmodes $\tilde{f}_k$, we can derive low entanglement, that is not dependent on the size of the system. Instead of giving a rigorous proof, I'd like to draw an intuitive picture.

While we have to give up local eigenmodes in a general system, entanglement is much easier to measure. Therefore showing little entanglement is an indicator of many body localization, but at this point a rigorous definition of many body localization has yet to be found.